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ON THE AVERAGE HYPEROSCILLATIONS OF PLANTED PLANE TREES

Peter KIRSCHENHOFER and Helmut PRODINGER

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Assume that the leaves of a planted plane tree are enumerated from left to right by 1, 2, ... The j-th s-turn of the tree is defined to be the root of the (unique) subtree of minimal height with leaves $j, j+1, \ldots, j+s-1$. If all trees with n nodes are regarded equally likely, the average level number of the j-th s-turn tends to a finite limit $\alpha_s(j)$, which is of order $j^{1/2}$. The j-th "s-hyperoscillation" $\alpha_1(j)-\alpha_{s+1}(j)$ is given by $\frac{1}{2}\,\alpha_1(s)+O(j^{-1/2})$ and therefore tends (for $j\to\infty$) to a constant behaving like $\sqrt{8/\pi}\cdot s^{1/2}$ for $s\to\infty$. These results are obtained by setting up appropriate generating functions, which are expanded about their (algebraic) singularities nearest to the origin, so that the asymptotic formulas are consequences of the so-called Darboux-Pólya-method.

1. Introduction

In a recent paper [11] R. Kemp studies the average oscillation of a stack during postorder traversing of a binary tree. The problem turns out to be equivalent with the study of the average level number of the so-called "MAX-turns" resp. "MIN-turns" of a planted plane tree, which are defined in the following way:

The MAX-turns are just the leaves of the tree (i.e. the nodes having no son), which are assumed to be enumerated from left to right by the natural numbers. The j-th MIN-turn is the root of the (uniquely determined) subtree of minimal height which has exactly the two leaves j and $j+1$. (Compare the examples in [11].) The average level number of the j-th MAX- (MIN-)turn of all planted plane trees with exactly n nodes (where all such trees are regarded equally likely) converges for $n\to\infty$ to a finite limit $\alpha_1(j)$ resp. $\alpha_2(j)$. (In Kemp's notation these numbers equal $\alpha_1(j)=\lim_{n\to\infty}\ell_n(j)$, $\alpha_2(j)=\lim_{n\to\infty}r_n(j)$.) Kemp shows that these numbers have the following asymptotic behaviour for $j\to\infty$: [11, Corollary 6]

$$\alpha_1(j)=\frac{8}{\sqrt{2\pi}}j^{1/2}+1+O(j^{-1/2}),$$

(1.1)

$$\alpha_2(j)=\frac{8}{\sqrt{2\pi}}j^{1/2}-1+O(j^{-1/2}).$$

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The difference

$$\alpha_1(j)-\alpha_2(j) = 2+O(j^{-1/2}) \tag{1.2}$$

describes the average oscillation of the contour of the trees.

Regarding these results, it is natural to consider the following more general problem: Define the j-th "s-turn" (for a tree with at least $j+s-1$ leaves) to be the root of the (uniquely determined) subtree of minimal height with leaves $j, j+1, \ldots, j+s-1$. The 1-turns (2-turns) are just the MAX-turns (MIN-turns) from above. Let $\alpha_s(j)$ denote the average level number of the j-th s-turn of all planted plane trees with exactly n nodes "for large n" (that means the limit of the mean value for $n\to\infty$). We will show that the differences $\alpha_1(j)-\alpha_{s+1}(j)$, which can be called the j-th "s-hyperoscillations" of the trees, behave like

$$\alpha_1(j)-\alpha_{s+1}(j) = \frac{\alpha_1(s)}{2}+O(j^{-1/2}), \quad (j\to\infty) \tag{1.3}$$

and therefore

$$\alpha_{s+1}(j) = \frac{8}{\sqrt{2\pi}}j^{1/2}-\varrho_s+O(j^{-1/2}), \quad (j\to\infty)$$

with

$$\varrho_s = \sqrt{\frac{8}{\pi}}\,s^{1/2}-\frac{1}{2}+O(s^{-1/2}), \quad (s\to\infty).$$

We would like to emphasize that the interest of this paper is largely methodological: Our way of handling the problem does *not* follow the classical approach of first deriving exact enumeration results and afterwards using approximations, but it heavily relies on the pure use of appropriate generating functions and the principle that the coefficients of a generating function are largely determined by the location and nature of its singularities. The advantage of this method is, that it will be applicable even in those cases where no exact enumeration formulas are available. (Compare e.g. [5], [13], [15], where a similar philosophy is underlying.)

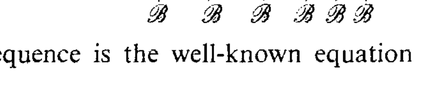
At the end of the paper we give an extensive list of papers dealing with related problems.

Remark. In the following we will frequently use the abbreviation $\langle f(x), x^i\rangle$ for the coefficient of x^i in the (formal) power series $f(x)$.

2. Generating functions

Using a suggestive terminology due to Ph. Flajolet [3] the family $\mathcal{B}$ of planted plane trees (sometimes also called "ordered trees") can be defined by the following formal equation.

(2.1)



An immediate consequence is the well-known equation

$$C(z) = z+zC(z)+zC^2(z)+\ldots = \frac{z}{1-C(z)} \tag{2.2}$$

for the generating function of the numbers $\langle C(z), z^n\rangle$ of planted plane trees with exactly n nodes (Catalan numbers).

For the sequel it is necessary to make some definitions: ("tree" will always denote a planted plane tree)

$t_s(n, \lambda)$ ... the number of trees with n nodes and λ s-turns

$$F_s(z, y) := \sum_{n\geqq 1}\sum_{\lambda\geqq 0} t_s(n,\lambda)z^n y^\lambda; \quad F(z, y) := F_1(z, y). \tag{2.3}$$

(That means $C(z)=F(z, 1)$.)

Relation (2.1) implies (compare [11])

$$F(z, y) = zy+\frac{zF(z, y)}{1-F(z, y)} = z(y-1)+\frac{z}{1-F(z, y)}. \tag{2.4}$$

Further we set

$$F_{s;j}(z, y) := \sum_{n\geqq 1}\sum_{\lambda\geqq j} t_s(n, \lambda)z^n y^\lambda, \quad C_{s;j} := F_{s;j}(z, 1) \tag{2.5}$$

for the corresponding generating functions of the trees with at least j s-turns.

Let $G_{s;j}^{[h]}(z)$ denote the generating function of the number of trees with at least j s-turns and a level number of the j-th s-turn which is greater than h ($\geqq 0$) and

$$H_{s;j}(z) := \sum_{h\geqq 0} G_{s;j}^{[h]}(z). \tag{2.6}$$

The coefficient of z^n in this last power series is just the sum of level numbers of the j-th s-turn. In order to derive a recurrence relation for the functions $H_{s;j}$ we first observe that the following holds:

Lemma 1. *For all* $h\geqq 0, j\geqq 1$

$$G_{s;j}^{[h+1]}(z) = z\sum_{t\geqq 1}\sum_{r=0}^{t-1} C(z)^{t-1-r}\sum_{i=0}^{j-1}\langle F(z, y)^r, y^i\rangle G_{s;j-i}^{[h]}(z). \tag{2.7}$$

Proof. Observe that by $h+1\geqq 1$ only those trees give a nonzero contribution to the series $G_{s;j}^{[h+1]}$, the j-th s-turn of which is *not* situated in the root. Let $r+1$ denote the number of that one of the t subtrees of the root, in which the j-th s-turn is situated. The first factor in the sum over r in (2.7) is constituted by the fact, that there is no restriction for the subtrees $r+2, \ldots, t$ (because they are situated on the right of the subtree in question). The sum-over-i-term originates from the observation, that the j-th s-turn of the tree is the $j-i$-th s-turn of the $r+1$-st subtree of the root iff the first r subtrees contain together i leaves. (If the j-th s-turn does not lie in the root, it must be situated in the same subtree as the j-th leaf.) ∎

Lemma 2. *For all* $j\geqq 1$

$$H_{s;j}(z) = C_{s;j}(z)+\frac{z}{1-C(z)}\sum_{i=0}^{j-1}\left\langle\frac{1}{1-F(z, y)}, y^i\right\rangle H_{s;j-i}(z). \tag{2.8}$$

Proof. Summing up in (2.7) and regarding that

$$G_{s;j}^{[0]}(z) = C_{s;j}(z)$$

yields

$$H_{s;j}(z) = C_{s;j}(z)+z \sum_{t\geqq 1} \sum_{r=0}^{t-1} C(z)^{t-1-r} \sum_{i=0}^{j-1} \langle F(z, y)^r, y^i\rangle H_{s;j-i}(z)$$

which equals

$$C_{s;j}(z)+z \sum_{i=0}^{j-1} H_{s;j-i}(z) \sum_{r\geqq 0} \langle F(z, y)^r, y^i\rangle \sum_{t-1\geqq r} C(z)^{t-1-r}$$

and this is just the right side of equation (2.8). ∎

The following considerations simplify very much by the use of the double generating function

$$H_s(z, y) := \sum_{j\geqq 1} H_{s;j}(z) y^j. \tag{2.9}$$

Lemma 3. $H_s(z, y) = \dfrac{y}{z(1-y)^2} \cdot \Delta(z, y) \cdot \Delta_s(z, y)$, *where*

$$\Delta(z,y) := C(z)-F(z,y), \quad \Delta_s(z,y) := C(z)-F_s(z,y). \tag{2.10}$$

Proof. By (2.8)

$$H_s(z, y) = \sum_{j\geqq 1} y^j C_{s;j}(z) + \frac{z}{1-C(z)} \frac{1}{1-F(z, y)} H_s(z, y).$$

Abel's summation formula shows that

$$\sum_{j\geqq 1} y^j C_{s;j}(z) = \sum_{j\geqq 1} [C_{s;j}(z)-C_{s;j+1}(z)] \frac{y^{j+1}-y}{y-1} =$$

$$= \sum_{j\geqq 1} \langle F_s(z, y), y^j\rangle \frac{y^{j+1}-y}{y-1} = \frac{y}{1-y} \Delta_s(z, y).$$

Relations (2.2) and (2.4) establish

$$\frac{z}{[1-C(z)][1-F(z,y)]} = \frac{1}{\Delta(z, y)} \left[\frac{z}{1-C(z)} - \frac{z}{1-F(z, y)}\right] =$$

$$= \frac{1}{\Delta(z, y)} [\Delta(z, y) - z(1-y)]$$

from which the result is immediate. ∎

The desired average level numbers $\alpha_s(j)$ of the j-th s-turn "for large n" are given by the limit

$$\alpha_s(j) = \lim_{n\to\infty} \frac{\langle H_{s;j}(z), z^n\rangle}{\langle C(z), z^n\rangle}. \tag{2.11}$$

In the following we will derive a theorem characterizing the generating functions

$$A_s(y) := \sum_{j\geqq 1} \alpha_s(j) y^j \tag{2.12}$$

of these numbers, the proof of which is established by investigating the behaviour of the functions $C(z)$ and $H_{s;j}(z)$ around their (algebraic) singularity $q\,(=1/4)$

nearest to the origin and does not depend on any knowledge of the explicit values of $t_s(n, \lambda)$.

By (2.2) we have

$$C(z) = \frac{1}{2}\left[1-\sqrt{1-4z}\right] = C(q)-(q-z)^{1/2}, \quad (z \to q^-) \tag{2.13}$$

where $q=\frac{1}{4}$ is the unique singularity of $C(z)$ on its circle of convergence.

A similar expansion holds for the functions $H_{s;j}(z)$:

Lemma 4.

$$H_{s;j}(z) = H_{s;j}(q)-a_{s,j}(q-z)^{1/2}+O((q-z)), \quad (z \to q^-) \tag{2.14}$$

where $q=\frac{1}{4}$ *is the unique singularity on the circle of convergence.*

Proof. Lemma 2 and Relations (2.2), (2.4) show that

$$H_{s;j}(z) = C_{s;j}(z)+C(z)\left[H_{s;j}(z)+\sum_{i=1}^{j-1}\left\langle\frac{F(z,y)}{z}+1-y, y^i\right\rangle H_{s;j-i}(z)\right]$$

or equivalently

$$H_{s;j}(z) = \frac{1}{z}C(z)C_{s;j}(z)+\frac{1}{z}C^2(z)\sum_{i=1}^{j-1}\left\langle\frac{F(z,y)}{z}-y, y^i\right\rangle H_{s;j-i}(z) \quad \text{for all} \quad j \geqq 1.$$

From this, by induction, it is sufficient for the proof of the lemma to show

a) that each of the functions $C_{s;j}(z)$ allows an expansion

$$C_{s;j}(z) = C_{s;j}(q)-(q-z)^{1/2}+O((q-z)), \quad (z \to q^-) \quad \text{and} \tag{2.15}$$

b) that for any fixed i the series $\langle F(z,y), y^i\rangle$ in z has a radius of convergence greater than q. (In fact it is 1.) Observing that a tree has j s-turns, iff it has $j+s-1$ leaves (for $j\geqq 1$; the case $j=0$ corresponds to a number of leaves in the interval $[1, s-1]$)

$$C_{s;j}(z) = C_{1;j+s-1}(z) = C(z)-\sum_{i=1}^{j+s-2}\langle F(z,y), y^i\rangle$$

and (2.15) is proved, if we prove assumption b). Now (2.4) shows

$$F(z,y) = \frac{1}{2}\left(1-z(1-y)-\sqrt{(1+z(1-y))^2-4z}\right), \tag{2.16}$$

which means, that for any fixed $0<y_0\leqq 1$ the series $F(z, y_0)$ has radius of convergence $(1+\sqrt{y_0})^{-2}$ (the term beyond the square root vanishes for $z=(1\pm\sqrt{y_0})^{-2}$). A fortiori, each of the coefficient series $\langle F(z,y), y^i\rangle$ must converge for $|z|<(1+\sqrt{y_0})^{-2}$ $(0<y_0\leqq 1)$, which means for any z with $|z|<1$ $(t_1(n,\lambda)\geqq 0!)$. ∎

By a well-known theorem of Darboux—Pólya (comp. [2, p. 277], [7, p. 211f], [15])

(2.17)
$$\langle C(z), z^n\rangle \sim \frac{1}{2}\sqrt{\frac{q}{\pi}}\, q^{-n} n^{-3/2}$$
$$\langle H_{s;j}(z), z^n\rangle \sim \frac{a_{s,j}}{2}\sqrt{\frac{q}{\pi}}\, q^{-n} n^{-3/2} \quad (n \to \infty)$$

and therefore

(2.18)
$$\alpha_s(j) = a_{s,j}.$$

This last relation may be expressed in a shorter form:

(2.19)
$$A_s(y) = \lim_{z\to q^-} \frac{H_s(q, y) - H_s(z, y)}{(q-z)^{1/2}}$$

where the limit shall be understood as to be carried out for each coefficient of the term, which is considered as a formal power series in y. Lemma 3 leads now to the following

Theorem 1. *The generating function of the average level numbers of the j-th s-turn* ($j \geqq 1$) *"for large n" is given by*

(2.20)
$$A_s(y) = \frac{4y}{(1-y)^2}\left(\Delta(q, y) + \Delta_s(q, y)\right).$$

Proof. Let $O_y(q-z)$ denote a formal power series in y, the coefficients $f_j(z)$ of which are functions in z behaving like $f_j(z) = O(q-z)$ for $z \to q^-$. (The O-constant may depend on j.) With this notion we have

(2.21)
$$\Delta(z, y) = \Delta(q, y) - (q-z)^{1/2} + O_y(q-z)$$
$$F_s(z, y) = F_s(q, y) + O_y(q-z)$$

by the same argument as in the proof of Lemma 4, and therefore

(2.22)
$$\Delta_s(z, y) = \Delta_s(q, y) - (q-z)^{1/2} + O_y(q-z).$$

Inserting (2.21) and (2.22) in Lemma 3 completes the proof. ∎

Observing

(2.23)
$$F(z, y) = y \cdot F_2(z, y)$$

the theorem allows to determine $A_1(y)$ and $A_2(y)$ immediately:

Corollary 1. *The generating function of the average level numbers of the j-th MAX-(MIN-)turn "for large n" is given by*

(2.24)
$$A_1(y) = \frac{1}{1-y} + \frac{2\sqrt{2}}{(1-y)^{3/2}}\left(1 + \frac{1-y}{8}\right)^{1/2}$$
$$A_2(y) = A_1(y) - \frac{2}{1-y} + \frac{1-y}{2y} \cdot A_1(y).$$

Proof. By (2.16) we obtain

$$\Delta(q, y) = C(q) - F(q, y) = \frac{1-y}{8} + \frac{(1-y)^{1/2}}{2\sqrt{2}}\left(1+\frac{1-y}{8}\right)^{1/2}, \tag{2.25}$$

thereby the first equation of (2.24). By (2.23)

$$\Delta_2(y) = C(q) - \frac{F(q, y)}{y}$$

and thus

$$\Delta(y) + \Delta_2(y) = \left(2+\frac{1-y}{y}\right)\Delta(y) - \frac{1-y}{y}C(q)$$

yielding the second identity of (2.24). ∎

So $A_1(y)$ and $A_2(y)$ have algebraic singularities at $y=1$, and the Darboux—Pólya-theorem establishes (1.1).

3. The average hyperoscillation

This section is devoted to the analysis of the asymptotic behaviour of the numbers $\alpha_s(j)$ for general s and $j \to \infty$. We start with the investigation of the functions $A_1(y) - A_{s+1}(y)$, that is with the generating functions of the j-th "s-hyperoscillations" $\alpha_1(j) - \alpha_{s+1}(j)$ (compare the comments in the introduction). The following theorem shows that there is an interesting symmetry in the behaviour of $\alpha_1(j) - \alpha_{s+1}(j)$ if considered as a function either of j (with fixed s) or of s (with fixed j).

Theorem 2. *The double generating function*

$$A(y, u) := \sum_{j\geq 1}\sum_{s\geq 1} y^j u^s \big(\alpha_1(j) - \alpha_{s+1}(j)\big) \tag{3.1}$$

of the j-th "s-hyperoscillation" fulfills

$$A(y, u) = \frac{4uy}{(1-y)(1-u)}\left[\frac{\Delta(q, u) - \Delta(q, y)}{y-u}\right]. \tag{3.2}$$

Proof. We will make use of the fact that, by the definitions,

$$F_s(z, y) = \frac{F_{1;s}(z, y)}{y^{s-1}} + [C(z) - C_{1;s}(z)]. \tag{3.3}$$

By Theorem 1

$$A(y, u) = \sum_{s\geq 1} u^s[A_1(y) - A_{s+1}(y)] = \frac{4y}{(1-y)^2}\sum_{s\geq 1} u^s[\Delta(q, y) - \Delta_{s+1}(q, y)] =$$

and with Abel's summation formula

$$= \frac{4yu}{(1-y)^2(1-u)}\left[\Delta(q, y) + \sum_{s\geq 1}(1-u^s)[\Delta_{s+2}(q, y) - \Delta_{s+1}(q, y)]\right] =$$

$$= \frac{4yu}{(1-y)^2(1-u)}\left[\Delta(q, y) - \Delta_2(q, y) - \sum_{s\geq 1} u^s[\Delta_{s+2}(q, y) - \Delta_{s+1}(q, y)]\right] =$$

which equals by (2.23) and (3.3)

$$= \frac{4yu}{(1-y)^2(1-u)}\left[-F(q,y)+\frac{F(q,y)}{y}+(1-y)\sum_{s\geq 1}u^s\frac{F_{1;s+1}(q,y)}{y^{s+1}}\right]=$$

$$= \frac{4yu}{(1-y)(1-u)}\left[\frac{1}{y}F(q,y)+\frac{1}{y}\sum_{s\geq 1}\left(\frac{u}{y}\right)^s F_{1;s+1}(q,y)\right]=$$

and again by Abel's summation formula

$$= \frac{4yu}{(1-y)(1-u)}\left[\frac{1}{y}F(q,y)+\frac{1}{y}\sum_{s\geq 1}\frac{\frac{u}{y}-\left(\frac{u}{y}\right)^{s+1}}{1-\frac{u}{y}}\times\right.$$

$$\left.\times[F_{1;s+1}(q,y)-F_{1;s+2}(q,y)]\right]=$$

$$= \frac{4yu}{(1-y)(1-u)}\left[\frac{1}{y}F(q,y)+\frac{1}{y}\frac{u}{y-u}F_{1;2}(q,y)-\right.$$

$$\left.-\frac{1}{y-u}\sum_{s\geq 1}u^{s+1}\langle F(q,y),y^{s+1}\rangle\right]=$$

$$= \frac{4yu}{(1-y)(1-u)}\left[\frac{1}{y}F(q,y)+\frac{1}{y}\frac{u}{y-u}[F(q,y)-y(C(q)-C_{1;2}(q))]-\right.$$

$$\left.-\frac{1}{y-u}[F(q,u)-u(C(q)-C_{1;2}(q))]\right]=$$

$$= \frac{4yu}{(1-y)(1-u)}\left[\frac{1}{y-u}F(q,y)-\frac{1}{y-u}F(q,u)\right]$$

from which (3.2) follows immediately. ∎

Theorem 2 allows to transfer the knowledge about the asymptotic behaviour of $\alpha_1(j), j\to\infty$, (Relation (1.1)) to the behaviour of $\alpha_{s+1}(j)$:

Corollary 2.

$$\alpha_{s+1}(j)=\alpha_1(j)-\frac{\alpha_1(s)}{2}+O(j^{-1/2}),\quad (j\to\infty) \tag{3.4}$$

or equivalently

$$\alpha_{s+1}(j)=\frac{8}{\sqrt{2\pi}}j^{1/2}-\varrho_s+O(j^{-1/2}),\quad (j\to\infty)$$

with

$$\varrho_s=\sqrt{\frac{8}{\pi}}s^{1/2}-\frac{1}{2}+O(s^{-1/2}),\quad (s\to\infty). \tag{3.5}$$

Proof. Using the notion $O_u(f(y))$ as defined in the proof of Theorem 1, we have by (3.2) and (2.25)

$$A(y, u) = \frac{4u}{(1-y)(1-u)}\left[\frac{\Delta(q, u)+O((1-y)^{1/2})}{(1-u)\left(1-\frac{1-y}{1-u}\right)}\right](1+O(1-y)) =$$

$$= \frac{4u\Delta(q, u)}{(1-y)(1-u)^2}+O_u((1-y)^{-1/2}),$$

$$= \frac{1}{1-y}\frac{A_1(u)}{2}+O_u((1-y)^{-1/2}), \quad (y \to 1-)$$

and therefore by Darboux's theorem

$$\alpha_1(j)-\alpha_{s+1}(j) = \frac{\alpha_1(s)}{2}+O(j^{-1/2}), \quad (j \to \infty).$$

Observing (1.1) the result is now immediate. ▮

Remark. The exact value of $\alpha_1(s)$ follows from Kemp [11, (37a)] resp. our generating function (2.24) to be

(3.6) $$\alpha_1(s) = \frac{4}{3}(s+2)-\frac{16}{9}S(s) \quad \text{where} \quad S(s) = \sum_{j=1}^{s-1}\sum_{k=1}^{j-1}\frac{1}{k(k+1)}3^{-k}P_k'\left(\frac{5}{3}\right)$$

(P_k' denotes the derivative of the k-th Legendre polynomial).

Corollary 2 implies

(3.7) $$\varrho_s = \frac{1}{3}+\frac{2}{3}s-\frac{8}{9}S(s).$$

The values computed by the asymptotic formula (3.5) coincide even for small s very well with the exact values computed by (3.6) and (3.7):

s	ϱ_s (exact value)	ϱ_s (asymptotic formula)
1	1.0000	1.0958
2	1.6667	1.7567
3	2.1852	2.2640
4	2.6214	2.6915
5	3.0046	3.0682
6	3.3502	3.4088
7	3.6673	3.7220
8	3.9622	4.0135
9	4.2387	4.2873
10	4.5000	4.5463

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Peter Kirschenhofer and Helmut Prodinger

Institut für Algebra und Diskrete Mathematik
TU Wien, Gußhausstraße 27—29,
A—1040 Wien, Austria